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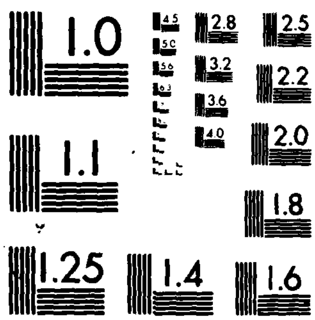

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CONSTRUCTION OF PBIB DESIGNS WITH TRIANGULAR AND L_2 SCHEMES

by

G.M. Constantine and A. Hedayat
Indiana University, Bloomington
and
University of Illinois, Chicago

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CONSTRUCTION OF PBIB DESIGNS WITH TRIANGULAR AND L_2 SCHEMES

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ABSTRACT

Simple methods of construction lead to families of PBIB designs with triangular and L_2 schemes. The construction can be carried out in s dimensions to obtain PBIB designs with at most s associate classes. A family of such PBIB designs (indexed by the block size) has the desirable statistical property that a pair of distinct varieties appears in at most one block.

Key words and phrases: PBIB design, triangular and L_2 association schemes, incomplete block design.

AMS 1970 subject classification: Primary 62K05; Secondary 62K10, 05B05.

1. Introduction.

An imcomplete block design is a k x b array with entries from the set of varieties $\{1,2,\ldots,v\}$, with $k < v$. The columns of the array are called blocks. An incomplete block design is called binary if all the blocks consist of distinct varieties.

A relation of association among the v varieties is said to be a triangular association scheme if $v = \binom{n}{2}$ and the v varieties appear as the upper-diagonal (and symmetrically as the lower-diagonal) entries of an n x n square array with the diagonal entries left blank. Two varieties are first associates if they lie in the same row or column; otherwise they are second associates.

An association scheme is called of the L_2 type if $v = n^2$ and the varieties can be arranged in a square in such a way that two varieties are first associates if they appear in the same row or column and second associates otherwise.

A triangular (respectively L_2) Partially Balanced Incomplete Block(PBIB) design with two associate classes is a binary incomplete block design in which the v varities form a traingular(respectively L_2) association scheme; first associates occur together in λ_1 blocks and second associates occur in λ_2 blocks of the design.

The PBIB designs have been developed by Bose and Nair(1939). Substantial work followed in classifying and analyzing these designs. Construction of triangular PBIB designs was done by Shrikhande(1952), Youden(1951), Bose and Shimamoto(1952), Chang, Liu and Liu(1965), Masuyama(1965), Liu and Chang(1964) and others. Among the constructions of L_2 PBIB designs we mention those by Yates(1936), Shrikhande(1959), Clatworthy(1967), Chang and Liu(1964) and Vartak(1955).

This paper presents easy and direct ways of constructing families of PBIB designs with triangular and L_2 association schemes. The method is applicable to higher dimensions; it yields PBIB designs with more than two associate classes.

In the case of triangular PBIB designs, the procedure described by Clatworthy(1956) with $v = \binom{n}{2}$, $b = \binom{n}{3}$, $r = n-2$, $k = 3$, $\lambda_1 = 1$ and $\lambda_2 = 0$ is a special case of our construction.

For every block size $k(\geq 2)$ we obtain a family of PBIB designs with parameters

$$v = \binom{n}{k-1}, \quad b = \binom{n}{k}, \quad k, \quad r = n-k+1$$

$$\lambda_1 = 1, \quad \lambda_2 = \lambda_3 = \ldots = \lambda_{k-1} = 0.$$

A design in this family has the property that any pair of distinct varieties appears in at most one block, a property which has partly been proved (and is generally believed) to ensure high statistical efficiency. Examples are intercalated to add to clarity.

For convenience we let $\binom{n}{m} = 0$ for m negative.

2. Construction

Let T be the n x n symmetrical array with its diagonal entries left blank and the $\binom{n}{2}$ varieties arranged in the upper (and lower) diagonal part of T.

Theorem 2.1. The distinct varieties common to m rows and the same m columns of T is defined to be a block. When all such choices of m rows and same m columns are considered we obtain a triangular PBIB design with $v = \binom{n}{2}$, $b = \binom{n}{m}$, $k = \binom{m}{2}$, $r = \binom{n-2}{m-2}$, $\lambda_1 = \binom{n-3}{m-3}$ and $\lambda_2 = \binom{n-4}{m-4}$.

These PBIB designs exist for all $2 \leq m \leq n$ and all $n \geq 4$.

Proof: Firstly observe that no matter what row (and same column) permutations are done to T, the incomplete block design that results (by applying the process described in the statement of the theorem to such

permuted T) is the same as the one that results from T itself. Any variety can be brought in position (1,2) in T by such row and column permutations. Any pair of distinct varieties that are in the same row or column can be put in positions (1,2) and (1,3) and finally any pair of distinct varieties not in the same row or column can be brought in positions (1,2) and (3,4). It is now easy to see that(following the process described in the theorem) we obtain a PBIB design with two associate classes, by just counting the number of blocks that contain varieties in these positions.

As an illustration, let $n = 5$ and $m = 3$. The triangular scheme is:

$$T = \begin{array}{ccccc} * & 1 & 2 & 3 & 4 \\ 1 & * & 5 & 6 & 7 \\ 2 & 5 & * & 8 & 9 \\ 3 & 6 & 8 & * & 10 \\ 4 & 7 & 9 & 10 & * \end{array}$$

and the resulting PBIB design with $v = 10$, $b = 10$, $k = 3$, $r = 3$, $\lambda_1 = 1$ and $\lambda_2 = 0$ is:

$$\begin{array}{cccccccccc} 1 & 1 & 1 & 2 & 2 & 3 & 5 & 5 & 6 & 8 \\ 2 & 3 & 4 & 3 & 4 & 4 & 6 & 7 & 7 & 9 \\ 5 & 6 & 7 & 8 & 9 & 10 & 8 & 9 & 10 & 10 \end{array} .$$

First associates of 1 are 2,3,4,5,6,7; second associates of 1 are 8,9 and 10.

There are exactly two subfamilies of designs in Theorem 2.1 for which λ_1 and λ_2 differ by one. The first is (for $m = 3$):

$$v = \binom{n}{2}, \quad b = \binom{n}{3}, \quad k = 3, \ r = n-2, \ \lambda_1 = 1, \ \lambda_2 = 0$$

and the second one is (for $m = n-1$):

$$v = \binom{n}{2}, \ b = n, \quad k = \binom{n-1}{2}, \quad r = n-2, \ \lambda_1 = 1, \ \lambda_2 = 0.$$

Denote by S the $n \times n$ square array consisting of n^2 varieties.

Theorem 2.2. Define a block to be the set of varieties common to m rows and (not necessarily same) m columns of S. When all choices of m rows and (independently) all choices of m columns are considered we obtain a PBIB design with an L_2 scheme and with $v = n^2$, $b = \binom{n}{m}^2$, $k = m^2$, $r = \binom{n-1}{m-1}^2$, $\lambda_1 = \binom{n-1}{m-1}\binom{n-2}{m-2}$ and $\lambda_2 = \binom{n-2}{m-2}^2$. Such PBIB designs exist for all $1 \le m \le n$ and all $n \ge 2$.

Proof: The procedure outlined in the statement of the theorem leads to the same incomplete block design when applied to any array obtained from S by (independent) row and column operations. With this observation it is easy to see that such a design will be a PBIB design with parameters as indicated in the theorem. This concludes our proof.

As an illustrative example, consider the simplest case with $n = 3$ and $m = 2$. The scheme is

$$S = \begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{matrix}$$

and the associated PBIB design is:

$$\begin{matrix} 1 & 1 & 2 & 1 & 1 & 2 & 4 & 4 & 5 \\ 2 & 3 & 5 & 7 & 3 & 8 & 5 & 6 & 8 \\ 4 & 4 & 3 & 2 & 7 & 3 & 7 & 7 & 6 \\ 5 & 6 & 6 & 8 & 9 & 9 & 8 & 9 & 9 \end{matrix} .$$

The parameters of this design are

$$v = 9, \quad b = 9, \quad k = 4, \quad r = 4, \quad \lambda_1 = 2 \quad \text{and} \quad \lambda_2 = 1.$$

This is the only design in the family of designs described in Theorem 2.2 for which λ_1 and λ_2 differ by one. First associates of 1 are 2,3,4,7 and second associates are 5,6,8,9.

3. Multidimensional extensions

(a) Theorem 2.1 can be extended to more than two dimensions in the following way. Initially identify each variety by an s-plet $(i_1, i_2, \ldots, i_s)$ where the symbols i_j satisfy $1 \le i_1 < i_2 < \ldots < i_s \le n$. We hence have $\binom{n}{s}$ varieties. Arrange the varieties (as s-plets) in lexicographical order. Then relabel them 1 through $\binom{n}{s}$.

We index the blocks by the subsets of size m of the set of the n symbols. For a subset of size m let the block be the collection of s-plets(varieties) that consist of only those symbols that belong to the subset. For this we of course need $s \le m \le n$. This construction yields $\binom{n}{m}$ blocks of size $\binom{m}{s}$. They form a PBIB design with parameters

$$v = \binom{n}{s}, \quad b = \binom{n}{m}, \quad k = \binom{m}{s}, \quad r = \binom{n-s}{m-s},$$

and

$$\lambda_i = \binom{n-s-i}{m-s-i} \qquad 1 \le i \le s .$$

It is easily seen that a design as above has a generalized triangular scheme with two varieties (as s-plets) being i^{th} associates if they differ in exactly i coordinates. These PBIB designs exist for all $s \le m \le n$.

Of particular importance is the case when $m = s+1(=k)$, for the reasons mentioned in the introduction. The parameters are:

$$v = \binom{n}{k-1}, \quad b = \binom{n}{k}, \quad k, \quad r = n-k+1$$
$$\lambda_1 = 1 \text{ and } \lambda_2 = \lambda_3 = \ldots = \lambda_{k-1} = 0.$$

We explicit this construction on an example. Let $s = 3$, $m = 4$ and $n = 6$. The varieties are:

1 = (123), 2 = (124), 3 = (125), 4 = (126), 5 = (134),
6 = (135), 7 = (136), 8 = (145), 9 = (146), 10 = (156),
11 = (234), 12 = (235), 13 = (236), 14 = (245), 15 = (246),
16 = (256), 17 = (345), 18 = (346), 19 = (356), 20 = (456).

index of blocks: $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 \\ 2 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 3 & 3 & 3 & 4 & 4 \\ 3 & 3 & 3 & 4 & 4 & 5 & 4 & 4 & 5 & 5 & 4 & 4 & 5 & 5 & 5 \\ 4 & 5 & 6 & 5 & 6 & 6 & 5 & 6 & 6 & 6 & 5 & 6 & 6 & 6 & 6 \end{pmatrix}$

PBIB design:

1	1	1	2	2	3	5	5	6	8	11	11	12	14	17
2	3	4	3	4	4	6	7	7	9	12	13	13	15	18
5	6	7	8	9	10	8	9	10	10	14	15	16	16	19
11	12	13	14	15	16	17	18	19	20	17	18	19	20	20

The parameters are:

$v = 20,\ b = 15,\ k = 4,\ r = 3,\ \lambda_1 = 1,\ \lambda_2 = 0,\ \lambda_3 = 0.$

The first associates of 1 are 2,3,4,5,6,7,11,12,13; second associates are 8,9,10,14,15,16,17,18,19 and 20 is the only third associate.

(b) The content of Theorem 2.2 can be extended as follows: Consider a hypercube C of dimension s. Place the n^s varieties as entries of this hypercube. Choose independently $m(< n)$ components along each one of the s coordinates. A block is defined as the collection of the m^s varieties whose coordinates belong to the chosen set of m components along each coordinate. When all the $\binom{n}{m}$ choices of components are independently considered along each coordinate we obtain a PBIB design with s associate classes with parameters.

$$v = n^s,\quad b = \binom{n}{m}^s,\ k = m^s\ ,\ r = \binom{n-1}{m-1}^s$$

and

$$\lambda_i = \binom{n-i}{m-i}^{s-i}\binom{n-i-1}{m-i-1}^i \qquad 1 \le i \le s.$$

Two varieties are i^{th} associates if they differ in exactly i coordinates as points of the hypercube. For any n, any s and any $1 \le m \le n - 1$ such PBIB designs can be constructed.

As an example, consider the case with s = 3, n = 3 and m = 2. We have 27 varieities and 27 blocks of size 8. By carrying out the construction we outlined we obtain the following PBIB design:

1	1	10	1	1	10	2	2	11
2	2	11	3	3	12	3	3	12
4	4	13	4	4	13	5	5	14
5	5	14	6	6	15	6	6	15
10	19	19	10	19	19	11	20	20
11	20	20	12	21	21	12	21	21
13	22	22	13	22	22	14	23	23
14	23	23	15	24	24	15	24	24

1	1	10	1	1	10	2	2	11
2	2	11	3	3	12	3	3	12
7	7	16	7	7	16	8	8	17
8	8	17	9	9	18	9	9	18
10	19	19	10	19	19	11	20	20
11	20	20	12	21	21	12	21	21
16	25	25	16	25	25	17	26	26
17	26	26	18	27	27	18	27	27

4	4	13	4	4	13	5	5	14
5	5	14	6	6	15	6	6	15
7	7	16	7	7	16	8	8	17
8	8	17	9	9	18	9	9	18
13	22	22	13	22	22	14	23	23
14	23	23	15	24	24	15	24	24
16	25	25	16	25	25	17	26	26
17	26	26	18	27	27	18	27	27

Variety 1 has 2,3,4,7,10,19 as first associates; 5,6,8,9,11,12,20,21,13, 16,22,25 as second associates and 14,15,17,18,23,24,26,27 as third associates. In this case $r = 8$, $\lambda_1 = 4$, $\lambda_2 = 2$ and $\lambda_3 = 1$.

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